

# An integrable semi-discrete Degasperis-Procesi equation

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**Abstract.** Based on our previous work to the Degasperis-Procesi equation (J. Phys. A **46** 045205) and the integrable semi-discrete analogue of its short wave limit (J. Phys. A **48** 135203), we derive an integrable semi-discrete Degasperis-Procesi equation by Hirota's bilinear method. Meanwhile,  $N$ -soliton solution to the semi-discrete Degasperis-Procesi equation is provided and proved. It is shown that the proposed semi-discrete Degasperis-Procesi equation, along with its  $N$ -soliton solution converge to ones of the original Degasperis-Procesi equation in the continuous limit.

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## 1. Introduction

In this paper, we are concerned with integrable discretization of the Degasperis-Procesi (DP) equation [1, 2]

$$m_t + 3mu_x + m_xu = 0, \quad m = -a + u - u_{xx}, \quad (1.1)$$

or

$$u_t - 3au_x - u_{txx} + 4uu_x = 3u_xu_{xx} + uu_{xxx}. \quad (1.2)$$

Prior to the DP equation, the Camassa-Holm (CH) equation firstly appeared in a mathematical search of recursion operators connected with the integrable partial differential equations [3] and then has attracted considerable attention since it was derived as a model equation for shallow water waves [4]. The CH and the DP equations are the only two integrable equations among the b-family equations [5]

$$m_t + bmu_x + m_xu = 0, \quad m = -a + u - u_{xx}, \quad (1.3)$$

or alternatively

$$u_t - bau_x - u_{txx} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \quad (1.4)$$

with  $b = 2$  and  $b = 3$ , respectively.

It is shown in [6] that both the CH and the DP equations can be used as models for the

propagation of shallow water waves over a flat bed, which accommodate wave breaking phenomena. These two equations are very similar, only differing by coefficients. They also share some common properties, for example, when  $a = 0$ , both the CH and DP equations have multi-peakon solutions [7, 8, 9] and admit wave breaking phenomena [10, 11].

On the other hand, although the DP equation has an apparent similarity to the CH equation, there are major structural differences between these two equations such as the Lax pair, blow up phenomena and the solutions. The isospectral problem in the Lax pair for the DP equation is the third-order equation [2], while the isospectral problem for the CH equation is the second order equation [4]. Therefore, the DP equation is much more complicated in term of the bi-Hamiltonian structures [12], the multi-peakon solution [13], the inverse scattering transform [14], the Riemann-Hilbert problem [15], as well as its bilinear equations and its multi-soliton solutions [16, 17, 18].

It is well-known that Hirota's bilinear method is very useful in finding multi-soliton solutions and constructing integrable discretizations to soliton equations [19]. Following this direction, the authors have succeeded in constructing integrable discretizations to a class of soliton equations with hodograph transformation. The examples include the short pulse equation and its multi-component generalizations [20, 21, 22, 23], the CH equation and its short wave limit (the Hunter-Saxton equation) [24, 25], the short wave limit of the DP equation (the reduced Ostrovsky equation) [26].

The goal of this paper is to construct integrable semi-discretization of the DP equation. It is a challenging problem in compared with the CH equation. In the present paper, we attempt to propose an integrable semi-discrete DP equation based on our previous results regarding the bilinear structure of the DP equation [18] and the integrable discretizations of its short wave limit [26]. The remainder of the present paper is organized as follows. In section 2, we gave a brief review for the bilinear equations with their connection with the negative flow of the CKP hierarchy. In section 3, semi-discrete integrable analogues of the bilinear equations of the DP equation are constructed in section 3. Then, we propose an integrable semi-discrete DP equation in section 4. The paper is concluded in section 5 by some comments and further topics.

## 2. Review for the bilinear equations of the Degasperis-Procesi equations and its $N$ -soliton solution

In this section, in order to make this paper self-contained, we will give a brief review to the bilinear equations of the Degasperis-Procesi equation, as well as its  $N$ -soliton solution as presented in [18].

We start with a sequence of  $2N \times 2N$  Gram-type determinants [27]

$$F_n = \det_{1 \leq i, j \leq 2N} (m_{ij}(n)), \quad (2.1)$$

where the entries of the determinant are defined by

$$m_{ij}(n) = \delta_{j, 2N+1-i} \frac{a^2}{2} \frac{\epsilon_{ij}(p_i - p_j)}{(1 + ap_i)(1 + ap_j)} + \frac{1}{p_i + p_j} \left( -\frac{p_i}{p_j} \right)^n \phi_i(0) \phi_j(0),$$

with

$$\varphi_i(k) = \left( \frac{1+ap_i}{1-ap_i} \right)^k e^{\xi_i}, \quad \xi_i = p_i^{-1}s + p_i y + \xi_{i0}, \quad \varepsilon_{ij} = \begin{cases} 1 & i < j \\ -1 & i > j \end{cases}.$$

Meanwhile a sequence of pfaffians which belongs to BKP hierarchy [28, 29] can be defined by

$$\tau_k = \text{Pf}(1, 2, \dots, 2N)_k \quad (2.2)$$

whose elements are

$$(i, j)_k = \delta_{j, 2N+1-i} \varepsilon_{ij} + \frac{p_i - p_j}{p_i + p_j} \varphi_i(k) \varphi_j(k).$$

Imposing a reduction condition

$$p_i^2 - p_i p_{2N+1-i} + p_{2N+1-i}^2 = \frac{1}{a^2},$$

and setting

$$f = \tau_0, \quad g = \tau_1, \quad F = F_0, \quad G = F_1, \quad (2.3)$$

as shown in [18], the following relations hold between the determinants and pfaffians

$$\left( D_y D_s - a D_y - \frac{1}{a} D_s \right) g \cdot f = 0, \quad (2.4)$$

$$gf = cG, \quad (2.5)$$

$$(-a D_y + 1) g \cdot f = cF, \quad (2.6)$$

$$\left( \frac{1}{2} D_y D_s - 1 \right) F \cdot F = -G^2, \quad (2.7)$$

where

$$c = \prod_{i=1}^{2N} 2p_i \frac{1+ap_i}{1-ap_i},$$

and  $D_y D_s$  is the Hirota  $D$ -operator defined by

$$D_y^m D_s^n f(y, s) \cdot g(y, s) = \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^m \left( \frac{\partial}{\partial s} - \frac{\partial}{\partial s'} \right)^n f(y, s) g(y', s')|_{y=y', s=s'}.$$

We comment here that the above four equations (2.4)–(2.7) are basically equivalent to Eqs. (2.20)–(2.24) in [18] although they seem slightly different. In what follows, we will show briefly how the equations (2.4)–(2.7) yield the DP equation through a dependent variable transformation

$$u = \left( \ln \frac{g}{f} \right)_s, \quad (2.8)$$

and a hodograph transformation

$$x = -\frac{1}{a}y + \ln \frac{g}{f}, \quad t = s. \quad (2.9)$$

Dividing  $gf$  on both sides, the first three equations (2.4)–(2.6) can be rewritten as

$$(\ln gf)_{ys} + \left( \left( \ln \frac{g}{f} \right)_y - \frac{1}{a} \right) \left( \left( \ln \frac{g}{f} \right)_s - a \right) - 1 = 0, \quad (2.10)$$

$$1 = \frac{cG}{gf}, \quad (2.11)$$

$$-a \left( \ln \frac{g}{f} \right)_y + 1 = \frac{cF}{gf}. \quad (2.12)$$

While, by dividing  $F^2$  on both sides, the bilinear equation (2.7) becomes

$$(\ln F)_{ys} - 1 = -\frac{G^2}{F^2}. \quad (2.13)$$

With the use of (2.11), (2.12) becomes

$$-a \left( \ln \frac{g}{f} \right)_y + 1 = \frac{F}{G}. \quad (2.14)$$

Subtracting (2.13) from (2.10), one obtains

$$\left( \ln \frac{G}{F} \right)_{ys} + \left( \left( \ln \frac{g}{f} \right)_y - \frac{1}{a} \right) \left( \left( \ln \frac{g}{f} \right)_s - a \right) = \frac{G^2}{F^2} \quad (2.15)$$

by referring to (2.11).

Introducing an intermediate variable  $\rho = G/F$ , one can calculate that

$$\frac{\partial x}{\partial y} = -\frac{1}{a} + \left( \ln \frac{g}{f} \right)_y = -\frac{1}{a\rho}, \quad \frac{\partial x}{\partial s} = \left( \ln \frac{g}{f} \right)_s = u,$$

based on the transformations (2.8)–(2.9), which yields a conversion formula

$$\partial_y = -\frac{1}{a\rho} \partial_x, \quad \partial_s = \partial_t + u \partial_x. \quad (2.16)$$

Substituting (2.14) into (2.15), one obtains

$$(\ln \rho)_{ys} - \frac{1}{a\rho} (u - a) = \rho^2, \quad (2.17)$$

which can be rewritten as

$$((\ln \rho)_s)_x + u - a = -a\rho^3. \quad (2.18)$$

On the other hand, differentiating (2.14) with respect to  $s$ , one yields

$$\left( \frac{1}{\rho} \right)_s + au_y = 0, \quad (2.19)$$

which, in turn, becomes

$$(\ln \rho)_s = -u_x \quad (2.20)$$

by using the conversion formula (2.16).

In the last, eliminating  $\rho$  from (2.18)–(2.20), one obtains

$$(\partial_t + u \partial_x) \ln(u - u_{xx} - a) = -3u_x, \quad (2.21)$$

which is nothing but the Degasperis-Procesi equation (1.2).

### 3. Semi-discrete analogue of equations (2.4)–(2.7)

Based on the results mentioned in the previous section, we attempt to construct an integrable semi-discrete analogue of the DP equation (1.2) by using Hirota's bilinear method. The key point is how to obtain discrete analogues of the equations (2.4)–(2.4) including the bilinear equations possessing  $N$ -soliton solutions.

Keeping in mind that the Degasperis-Procesi equation is derived from pseudo 3-reduction of the CKP hierarchy, we start with the Gram-type determinants, which are soliton solutions of the CKP hierarchy,

$$F_{k,l} = \det_{1 \leq i,j \leq 2N} (m_{ij}(k,l)), \quad G_{k,l} = \det_{1 \leq i,j \leq 2N} (m'_{ij}(k,l)),$$

where

$$m_{ij}(k,l) = C_{ij} + \frac{1}{p_i + p_j} \phi_i^{(0)}(k,l) \phi_j^{(0)}(k,l),$$

$$m'_{ij}(k,l) = C_{ij} + \frac{1}{p_i + p_j} \left( -\frac{p_j}{p_i} \right) \frac{1 + bp_i}{1 - bp_j} \phi_i^{(0)}(k,l) \phi_j^{(0)}(k,l),$$

with

$$C_{ij} = C_{ji}, \quad \phi_i^{(n)}(k,l) = p_i^n \left( \frac{1 + ap_i}{1 - ap_i} \right)^k \left( \frac{1 + bp_i}{1 - bp_i} \right)^l e^{\xi_i}, \quad \xi_i = p_i^{-1}s + \xi_{i0}.$$

Here  $2b$  (not  $b$ ) is the mesh size in  $y$ -direction. The relation between  $F_{k,l}$  and  $G_{k,l}$  is given by the following lemma.

**Lemma 3.1.**

$$(D_s - 2b)F_{k,l+1} \cdot F_{k,l} = -2bG_{k,l}^2, \quad (3.1)$$

**Proof.** It can be easily verified that

$$\partial_s m_{ij}(k,l) = \phi_i^{(-1)}(k,l) \phi_j^{(-1)}(k,l),$$

$$m_{ij}(k,l+1) = m_{ij}(k,l) + \frac{2b}{(1 - bp_i)(1 - bp_j)} \phi_i^{(0)}(k,l) \phi_j^{(0)}(k,l),$$

and

$$m'_{ij}(k,l) = m_{ij}(k,l) - \frac{1}{1 - bp_j} \phi_i^{(-1)}(k,l) \phi_j^{(0)}(k,l).$$

Then we have

$$\partial_s F_{k,l} = \begin{vmatrix} m_{ij}(k,l) & \phi_i^{(-1)}(k,l) \\ -\phi_j^{(-1)}(k,l) & 0 \end{vmatrix},$$

$$F_{k,l+1} = \begin{vmatrix} m_{ij}(k,l) & \frac{2b}{1 - bp_i} \phi_i^{(0)}(k,l) \\ -\frac{1}{1 - bp_j} \phi_j^{(0)}(k,l) & 1 \end{vmatrix},$$

$$G_{k,l} = \begin{vmatrix} m_{ij}(k,l) & \phi_i^{(-1)}(k,l) \\ \frac{1}{1-bp_j}\phi_j^{(0)}(k,l) & 1 \end{vmatrix} = \begin{vmatrix} m_{ij}(k,l) & \frac{1}{1-bp_i}\phi_i^{(0)}(k,l) \\ \phi_j^{(-1)}(k,l) & 1 \end{vmatrix},$$

$$(\partial_s - 2b)F_{k,l+1} = \begin{vmatrix} m_{ij}(k,l) & \phi_i^{(-1)}(k,l) & \frac{2b}{1-bp_i}\phi_i^{(0)}(k,l) \\ -\phi_j^{(-1)}(k,l) & 0 & -2b \\ -\frac{1}{1-bp_j}\phi_j^{(0)}(k,l) & -1 & 1 \end{vmatrix}.$$

By the Jacobi identity of determinants, we obtain

$$(\partial_s - 2b)F_{k,l+1} \times F_{k,l} = F_{k,l+1} \times \partial_s F_{k,l} - (-2bG_{k,l}) \times (-G_{k,l}),$$

which is exactly the bilinear equation (3.1).  $\square$

Next, we perform reductions similar to the pseudo 3-reduction of the CKP hierarchy in the continuous case. To this end, we take

$$C_{ij} = \delta_{j,2N+1-i}c_i, \quad c_{2N+1-i} = c_i, \quad (3.2)$$

and further assume

$$c_{ij} = 2p_i C_{ij} \frac{1+ap_i}{1-ap_j} \frac{1-bp_j}{1+bp_i}. \quad (3.3)$$

By imposing a reduction condition

$$\frac{(1-a^2p_{2N+1-i}^2)(1-b^2p_i^2)}{p_i} + \frac{(1-a^2p_i^2)(1-b^2p_{2N+1-i}^2)}{p_{2N+1-i}} = 0, \quad (3.4)$$

or, equivalently,

$$\frac{p_i(1+ap_i)(1-bp_{2N+1-i})}{(1-ap_{2N+1-i})(1+bp_i)} = -\frac{p_{2N+1-i}(1+ap_{2N+1-i})(1-bp_i)}{(1-ap_i)(1+bp_{2N+1-i})}, \quad (3.5)$$

it then follows that

$$\begin{aligned} c_{ij} &= \delta_{j,2N+1-i}c_i \frac{2p_i(1+ap_i)}{1-ap_{2N+1-i}} \frac{1-bp_{2N+1-i}}{1+bp_i} \\ &= -\delta_{j,2N+1-i}c_{2N+1-i} \frac{2p_{2N+1-i}(1+ap_{2N+1-i})}{1-ap_i} \frac{1-bp_i}{1+bp_{2N+1-i}} \\ &= -c_{ji}. \end{aligned}$$

Therefore, we can define a pfaffian of the form

$$f_{kl} = \text{Pf}(1, 2, \dots, 2N)_{kl},$$

whose elements defined by

$$(i, j)_{kl} = c_{ij} + \frac{p_i - p_j}{p_i + p_j} \phi_i^{(0)}(k, l) \phi_j^{(0)}(k, l).$$

Then the following lemma provides a bilinear equation satisfied by the pfaffian  $f_{kl}$ .

**Lemma 3.2.**

$$\left( \frac{1}{a+b} D_s - 1 \right) f_{k+1, l+1} \cdot f_{kl} = \left( \frac{1}{a-b} D_s - 1 \right) f_{k+1, l} \cdot f_{k, l+1}, \quad (3.6)$$

**Proof.** Letting

$$\begin{aligned}(i, d_n)_{kl} &= \varphi_i^{(n)}(k, l), \quad (d_m, d_n)_{kl} = 0, \\ (i, d^k)_{kl} &= \varphi_i^{(0)}(k+1, l), \quad (d_0, d^k)_{kl} = 1, \quad (d_{-1}, d^k)_{kl} = -a, \\ (i, d^l)_{kl} &= \varphi_i^{(0)}(k, l+1), \quad (d_0, d^l)_{kl} = 1, \quad (d_{-1}, d^l)_{kl} = -b,\end{aligned}$$

and

$$(d^l, d^k)_{kl} = \frac{a-b}{a+b},$$

it is shown in Appendix that

$$\partial_s f_{kl} = (1, 2, \dots, 2N, d_{-1}, d_0)_{kl}, \quad (3.7)$$

$$f_{k+1, l} = (1, 2, \dots, 2N, d_0, d^k)_{kl}, \quad (3.8)$$

$$f_{k, l+1} = (1, 2, \dots, 2N, d_0, d^l)_{kl}, \quad (3.9)$$

$$(\partial_s - a)f_{k+1, l} = (1, 2, \dots, 2N, d_{-1}, d^k)_{kl}, \quad (3.10)$$

$$(\partial_s - b)f_{k, l+1} = (1, 2, \dots, 2N, d_{-1}, d^l)_{kl}, \quad (3.11)$$

$$\frac{a-b}{a+b}f_{k+1, l+1} = (1, 2, \dots, 2N, d^l, d^k)_{kl}, \quad (3.12)$$

$$(\partial_s - a - b)\frac{a-b}{a+b}f_{k+1, l+1} = (1, 2, \dots, 2N, d_{-1}, d_0, d^l, d^k)_{kl}. \quad (3.13)$$

Therefore, an algebraic identity of pfaffian [19]

$$\begin{aligned}\text{Pf}(\dots, d_{-1}, d_0, d^l, d^k)\text{Pf}(\dots) &= \text{Pf}(\dots, d_{-1}, d_0)\text{Pf}(\dots, d^l, d^k) \\ &\quad - \text{Pf}(\dots, d_{-1}, d^l)\text{Pf}(\dots, d_0, d^k) + \text{Pf}(\dots, d_{-1}, d^k)\text{Pf}(\dots, d_0, d^l),\end{aligned}$$

implies

$$\begin{aligned}(\partial_s - a - b)\frac{a-b}{a+b}f_{k+1, l+1} &\times f_{kl} \\ &= \frac{a-b}{a+b}f_{k+1, l+1} \times \partial_s f_{kl} - f_{k+1, l} \times (\partial_s - b)f_{k, l+1} + (\partial_s - a)f_{k+1, l} \times f_{k, l+1},\end{aligned}$$

which is nothing but the bilinear equation (3.6).  $\square$

The lemma below states the relations between pfaffian and determinant defined previously

**Lemma 3.3.**

$$f_{k+1, l}f_{kl} = c'G_{kl}, \quad (3.14)$$

$$(a-b)f_{k+1, l+1}f_{kl} - (a+b)f_{k+1, l}f_{k, l+1} = -2bc'F_{k, l+1}, \quad (3.15)$$

with

$$c' = \prod_{i=1}^{2N} 2p_i \frac{1+ap_i}{1-ap_i} \frac{1-bp_i}{1+bp_i}.$$

**Proof.** By an identity of pfaffian [18]

$$\begin{aligned}
f_{k+1,l}f_{kl} &= \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k+1,l) \\ & 1 & \\ & & 1 \end{pmatrix} \times \text{Pf}((i,j)_{kl}) \\
&= \begin{vmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) \\ \varphi_j^{(0)}(k+1,l) & 1 \end{vmatrix} \\
&= \det_{1 \leq i,j \leq 2N} \left( (i,j)_{kl} - \varphi_i^{(0)}(k,l)\varphi_j^{(0)}(k+1,l) \right) \\
&= \det \left( c_{ij} + \frac{p_i - p_j}{p_i + p_j} \varphi_i^{(0)}(k,l)\varphi_j^{(0)}(k,l) - \varphi_i^{(0)}(k,l)\varphi_j^{(0)}(k+1,l) \right) \\
&= \det \left( c_{ij} + \frac{-2p_j}{p_i + p_j} \frac{1 + ap_i}{1 - ap_j} \varphi_i^{(0)}(k,l)\varphi_j^{(0)}(k,l) \right) \\
&= c' \det \left( c_{ij} \frac{1}{2p_i} \frac{1 - ap_j}{1 + ap_i} \frac{1 + bp_i}{1 - bp_j} + \frac{1}{p_i + p_j} \left( -\frac{p_j}{p_i} \right) \frac{1 + bp_i}{1 - bp_j} \varphi_i^{(0)}(k,l)\varphi_j^{(0)}(k,l) \right) \\
&= c' \det \left( C_{ij} + \frac{1}{p_i + p_j} \left( -\frac{p_j}{p_i} \right) \frac{1 + bp_i}{1 - bp_j} \varphi_i^{(0)}(k,l)\varphi_j^{(0)}(k,l) \right) \\
&= c' \det \left( m'_{ij}(k,l) \right). \tag{3.16}
\end{aligned}$$

Thus, Eq. (3.14) is proved. Next, we proceed to the proof of Eq. (3.15). Firstly, by the same identity as above, the products of pfaffians can be rewritten into determinants

$$\begin{aligned}
\frac{a-b}{a+b} f_{k+1,l+1} f_{kl} &= \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l) \\ & \frac{a-b}{a+b} & \\ & & 1 \end{pmatrix} \times \text{Pf}((i,j)_{kl}) \\
&= \begin{vmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l+1) \\ \varphi_j^{(0)}(k+1,l) & \frac{a-b}{a+b} \end{vmatrix}, \tag{3.17}
\end{aligned}$$

$$\begin{aligned}
f_{k+1,l} f_{k,l+1} &= \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k+1,l) \\ & 1 & \\ & & 1 \end{pmatrix} \times \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) \\ & 1 & \\ & & 1 \end{pmatrix} \\
&= \begin{vmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) \\ -\varphi_j^{(0)}(k,l) & 0 & 1 \\ -\varphi_j^{(0)}(k+1,l) & -1 & -\frac{a-b}{a+b} \end{vmatrix}. \tag{3.18}
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{a-b}{a+b} f_{k+1,l+1} f_{kl} - f_{k+1,l} f_{k,l+1} \\
&= \begin{vmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l+1) \\ \varphi_j^{(0)}(k+1,l) & \frac{a-b}{a+b} \end{vmatrix} - \begin{vmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) \\ -\varphi_j^{(0)}(k,l) & 0 & 1 \\ -\varphi_j^{(0)}(k+1,l) & -1 & -\frac{a-b}{a+b} \end{vmatrix}
\end{aligned}$$



$$\begin{aligned}
&= \begin{vmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) \\ -\varphi_j^{(0)}(k,l) & 1 & 1 \\ \varphi_j^{(0)}(k+1,l) & 1 & \frac{a-b}{a+b} \end{vmatrix} \\
&= \begin{vmatrix} (i,j)_{kl} + \varphi_i^{(0)}(k,l)\varphi_j^{(0)}(k,l) & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) - \varphi_i^{(0)}(k,l) \\ 0 & 1 & 0 \\ \varphi_j^{(0)}(k+1,l) + \varphi_j^{(0)}(k,l) & 1 & \frac{-2b}{a+b} \end{vmatrix} \\
&= \begin{vmatrix} (i,j)_{kl} + \varphi_i^{(0)}(k,l)\varphi_j^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) - \varphi_i^{(0)}(k,l) \\ \varphi_j^{(0)}(k+1,l) + \varphi_j^{(0)}(k,l) & \frac{-2b}{a+b} \end{vmatrix} \\
&= \frac{-2b}{a+b} \det \left( c_{ij} + \frac{2p_i(1+ap_i)(1+bp_j)}{(p_i+p_j)(1-bp_i)(1-ap_j)} \varphi_i^{(0)}(k,l)\varphi_j^{(0)}(k,l) \right) \\
&= \frac{-2b}{a+b} \det \left( c_{ij} + \frac{2p_i(1+ap_i)(1-bp_j)}{(p_i+p_j)(1+bp_i)(1-ap_j)} \varphi_i^{(0)}(k,l+1)\varphi_j^{(0)}(k,l+1) \right) \\
&= \frac{-2bc'}{a+b} \det \left( c_{ij} \frac{1}{2p_i} \frac{1-ap_j}{1+ap_i} \frac{1+bp_i}{1-bp_j} + \frac{1}{p_i+p_j} \varphi_i^{(0)}(k,l+1)\varphi_j^{(0)}(k,l+1) \right) \\
&= \frac{-2bc'}{a+b} \det \left( m_{ij}(k,l+1) \right). \tag{3.19}
\end{aligned}$$

Multiplying both sides by  $(a+b)$ , we arrive at (3.15).  $\square$

Summarizing what we have discussed and letting  $f_l = f_{0l}$ ,  $g_l = f_{1l}$ ,  $F_l = F_{0l}$ ,  $G_l = G_{0l}$ , we arrive at the following four equations

$$\left( \frac{1}{a+b} D_s - 1 \right) g_{l+1} \cdot f_l = \left( \frac{1}{a-b} D_s - 1 \right) g_l \cdot f_{l+1}, \tag{3.20}$$

$$g_l f_l = c' G_l, \tag{3.21}$$

$$(D_s - 2b) F_{l+1} \cdot F_l = -2b G_l^2, \tag{3.22}$$

$$(a-b) g_{l+1} f_l - (a+b) g_l f_{l+1} = -2bc' F_{l+1}. \tag{3.23}$$

In fact, Eqs. (3.20)–(3.23) are integrable semi-discrete analogues of Eqs. (2.4)–(2.7). In other words, in the limit of  $b \rightarrow 0$ , Eqs. (3.20)–(3.23) converge to Eqs. (2.4)–(2.7), respectively. Meanwhile, the pfaffian and determinant solutions satisfying Eqs. (3.20)–(3.23) also converge to the pfaffian and determinant solutions satisfying Eqs. (2.4)–(2.7).

Note that  $2b$  is the mesh size. In the limit of  $b \rightarrow 0$ , we have  $c' \rightarrow c$ ,

$$f_l \rightarrow f, \quad g_l \rightarrow g, \quad f_{l+1} \rightarrow f + 2bf_y, \quad g_{l+1} \rightarrow g + 2bg_y,$$

and similar relations for the determinants  $F_l$ ,  $G_l$ ,  $F_{l+1}$  and  $G_{l+1}$ . Obviously, (3.21) goes to (2.5) as  $b \rightarrow 0$ . It can be easily shown that

$$\frac{1}{2b} D_s F_{l+1} \cdot F_l \rightarrow \frac{1}{2} D_s D_y F \cdot F,$$

and

$$\frac{1}{2b} (g_{l+1} f_l - g_l f_{l+1}) \rightarrow D_y g \cdot f, \quad \frac{1}{2} (g_{l+1} f_l + g_l f_{l+1}) \rightarrow gf.$$

Therefore, by dividing  $2b$  on both sides, (3.22) and (3.23) converge to (2.6) and (2.7), respectively, as  $b \rightarrow 0$ . Now we show the convergence of the first bilinear equation. It is obvious by noting that

$$\begin{aligned}
& \frac{a}{2b} \left( \frac{1}{a+b} D_s g_{l+1} \cdot f_l - \frac{1}{a-b} D_s g_l \cdot f_{l+1} \right) \\
& \rightarrow \frac{1}{2b} \left( 1 - \frac{b}{a} \right) D_s g_{l+1} \cdot f_l - \left( 1 + \frac{b}{a} \right) D_s g_l \cdot f_{l+1} \\
& \rightarrow \frac{1}{2b} D_s (g_{l+1} \cdot f_l - g_l \cdot f_{l+1}) - \frac{1}{2a} D_s (g_{l+1} \cdot f_l + g_l \cdot f_{l+1}) \\
& \rightarrow \frac{1}{2} D_s D_y g \cdot f - \frac{1}{a} g f,
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
& \frac{a}{2b} (g_{l+1} \cdot f_l - g_l \cdot f_{l+1}) \\
& \rightarrow a D_y g \cdot f.
\end{aligned} \tag{3.25}$$

#### 4. Semi-discrete Degasperis-Procesi equation

Now that we have constructed integrable semi-discrete analogues (3.20)–(3.23) of a set of equations (2.4)–(2.7) which derive the Degasperis-Procesi equation, we proceed to construct an integrable semi-discrete Degasperis-Procesi equation based on Hirota's bilinear method. First, let us work on the bilinear equation (3.20), which can be recast into

$$2g_{l+1}f_l \left( (a-b) \left( \ln \frac{g_{l+1}}{f_l} \right)_s - a^2 + b^2 \right) - 2g_l f_{l+1} \left( (a+b) \left( \ln \frac{g_l}{f_{l+1}} \right)_s - a^2 + b^2 \right) = 0,$$

by multiplying  $2(a^2 - b^2)$  on both sides, or,

$$\begin{aligned}
& ((a-b)g_{l+1}f_l + (a+b)g_l f_{l+1}) \left( \ln \frac{g_{l+1}f_{l+1}}{g_l f_l} \right)_s \\
& + ((a-b)g_{l+1}f_l - (a+b)g_l f_{l+1}) \left( \ln \frac{g_{l+1}g_l}{f_{l+1}f_l} \right)_s - 2(a^2 - b^2)(g_{l+1}f_l - g_l f_{l+1}) = 0.
\end{aligned}$$

Rearranging the terms, one obtains

$$\begin{aligned}
& ((a-b)g_{l+1}f_l + (a+b)g_l f_{l+1}) \left( \ln \frac{g_{l+1}f_{l+1}}{g_l f_l} \right)_s - 2b((a-b)g_{l+1}f_l + (a+b)g_l f_{l+1}) \\
& + ((a-b)g_{l+1}f_l - (a+b)g_l f_{l+1}) \left( \left( \ln \frac{g_{l+1}g_l}{f_{l+1}f_l} \right)_s - 2a \right) = 0.
\end{aligned} \tag{4.1}$$

Dividing  $((a-b)g_{l+1}f_l + (a+b)g_l f_{l+1})$  on both sides of (4.1), we have

$$\left( \ln \frac{g_{l+1}f_{l+1}}{g_l f_l} \right)_s + \frac{(a-b)g_{l+1}f_l - (a+b)g_l f_{l+1}}{(a-b)g_{l+1}f_l + (a+b)g_l f_{l+1}} \left( \left( \ln \frac{g_{l+1}g_l}{f_{l+1}f_l} \right)_s - 2a \right) - 2b = 0. \tag{4.2}$$

Secondly, Eqs. (3.22) and (3.23) can be easily rewritten as

$$\frac{(a-b)g_{l+1}f_l - (a+b)g_l f_{l+1}}{(a-b)g_{l+1}f_l + (a+b)g_l f_{l+1}} = -\frac{2bc'F_{l+1}}{(a-b)g_{l+1}f_l + (a+b)g_l f_{l+1}}, \tag{4.3}$$

and

$$\left( \ln \frac{F_{l+1}}{F_l} \right)_s - 2b = -2b \frac{G_l^2}{F_{l+1}F_l}, \quad (4.4)$$

respectively. Subtracting Eq. (4.4) from Eq. (4.2), we get

$$\left( \ln \frac{G_{l+1}F_l}{G_lF_{l+1}} \right)_s + \frac{(a-b)g_{l+1}f_l - (a+b)g_l f_{l+1}}{(a-b)g_{l+1}f_l + (a+b)g_l f_{l+1}} \left( \left( \ln \frac{g_{l+1}g_l}{f_{l+1}f_l} \right)_s - 2a \right) = 2b \frac{G_l^2}{F_{l+1}F_l} \quad (4.5)$$

by referring to Eq. (3.21).

Introducing variable transformations

$$u_l = \left( \ln \frac{g_l}{f_l} \right)_s, \quad r_l = \frac{G_l}{F_l}, \quad (4.6)$$

and a discrete hodograph transformation

$$\delta_l = -\frac{4bc'F_{l+1}}{(a-b)g_{l+1}f_l + (a+b)g_l f_{l+1}}, \quad t = s, \quad (4.7)$$

we then have

$$\frac{(a-b)g_{l+1}f_l - (a+b)g_l f_{l+1}}{(a-b)g_{l+1}f_l + (a+b)g_l f_{l+1}} = \frac{\delta_l}{2} \quad (4.8)$$

from Eq. (4.3). A substitution of Eq. (4.8) into Eq. (4.5) leads to

$$\left( \ln \frac{r_{l+1}}{r_l} \right)_s + \delta_l \left( \frac{u_{l+1} + u_l}{2} - a \right) = 2b \frac{F_l}{F_{l+1}} r_l^2 \quad (4.9)$$

by referring variable transformations.

Furthermore, based on (4.3) and (4.7), we obtain

$$-\frac{a-b}{b} \frac{g_{l+1}f_l}{c'F_{l+1}} = \frac{2}{\delta_l} + 1, \quad -\frac{a+b}{b} \frac{g_l f_{l+1}}{c'F_{l+1}} = \frac{2}{\delta_l} - 1. \quad (4.10)$$

Multiplying above two equations leads to

$$\frac{a^2 - b^2}{b^2} \frac{F_l}{F_{l+1}} r_{l+1} r_l = \frac{4}{\delta_l^2} - 1, \quad (4.11)$$

while dividing them yields

$$\frac{a-b}{a+b} \frac{g_{l+1}f_l}{f_{l+1}g_l} = \frac{2 + \delta_l}{2 - \delta_l}. \quad (4.12)$$

Taking logarithmic differentiation of Eq. (4.11) and Eq. (4.12) with respect to  $s$ , one obtains

$$(\ln r_{l+1} r_l)_s - \left( \ln \frac{F_{l+1}}{F_l} \right)_s = \frac{-8}{(4 - \delta_l^2) \delta_l} \frac{d\delta_l}{ds}, \quad (4.13)$$

and

$$u_{l+1} - u_l = \frac{4}{4 - \delta_l^2} \frac{d\delta_l}{ds}, \quad (4.14)$$

respectively. Eq. (4.14) can be rewritten as

$$\frac{d\delta_l}{ds} = \left( 1 - \frac{\delta_l^2}{4} \right) (u_{l+1} - u_l) \quad (4.15)$$

which constitutes one of the semi-discrete DP equation, describing the time evolution of the nonuniform mesh. Eliminating  $d\delta_l/ds$  from Eqs. (4.13) and (4.14), one obtains

$$\begin{aligned}
\frac{u_{l+1} - u_l}{\delta_l} &= -\frac{1}{2}(\ln r_{l+1} r_l)_s + \frac{1}{2} \left( \ln \frac{F_{l+1}}{F_l} \right)_s \\
&= -\frac{1}{2}(\ln r_{l+1} r_l)_s + b - b \frac{G_l^2}{F_{l+1} F_l} \\
&= -\frac{1}{2}(\ln r_{l+1} r_l)_s + b - b \frac{F_l}{F_{l+1}} r_l^2 \\
&= -\frac{1}{2}(\ln r_{l+1} r_l)_s + b - \frac{1}{2} \left( \ln \frac{r_{l+1}}{r_l} \right)_s - \frac{\delta_l}{2} \left( \frac{u_{l+1} + u_l}{2} - a \right) \\
&= -(\ln r_{l+1})_s + b - \frac{\delta_l}{2} \left( \frac{u_{l+1} + u_l}{2} - a \right). \tag{4.16}
\end{aligned}$$

Here Eqs. (4.4) and (4.9) are used.

In summary, we propose an integrable semi-discrete Degasperis-Procesi equation

$$\frac{1}{\delta_l} \left( \ln \frac{r_{l+1}}{r_l} \right)_s + \frac{u_{l+1} + u_l}{2} - a = \frac{2b}{\delta_l} \frac{r_l}{r_{l+1}} \frac{\frac{4}{\delta_l^2} - 1}{\frac{a^2}{b^2} - 1}, \tag{4.17}$$

$$(\ln r_{l+1})_s = -\frac{u_{l+1} - u_l}{\delta_l} + b - \frac{\delta_l}{2} \left( \frac{u_{l+1} + u_l}{2} - a \right), \tag{4.18}$$

$$\frac{d\delta_l}{ds} = \left( 1 - \frac{\delta_l^2}{4} \right) (u_{l+1} - u_l), \tag{4.19}$$

where an intermediate variable  $r_l$  is used.

**Remark 4.1.** Due to the fact

$$\frac{\delta_l}{2b} = \frac{x_{l+1} - x_l}{2b} \rightarrow \frac{\partial x}{\partial y} = -\frac{1}{ar},$$

as  $b \rightarrow 0$ , it is obvious that Eqs. (4.17) (or 4.5) and (4.18) converge to Eqs. (2.17) and (2.19), respectively.

In order to eliminate the intermediate variable  $r_l$ , we substitute Eq. (4.18) into Eq. (4.17) and get

$$\frac{u_l - u_{l-1}}{\delta_{l-1}} + \frac{\delta_{l-1}}{2} \left( \frac{u_l + u_{l-1}}{2} - a \right) - \frac{u_{l+1} - u_l}{\delta_l} + \frac{\delta_l}{2} \left( \frac{u_{l+1} + u_l}{2} - a \right) = \frac{2b r_l}{r_{l+1}} \frac{\frac{4}{\delta_l^2} - 1}{\frac{a^2}{b^2} - 1}. \tag{4.20}$$

Defining

$$m_l = \frac{2}{\delta_l + \delta_{l-1}} \left( -\frac{u_{l+1} - u_l}{\delta_l} + \frac{u_l - u_{l-1}}{\delta_{l-1}} + \frac{\delta_l(u_{l+1} + u_l) + \delta_{l-1}(u_l + u_{l-1})}{4} \right) - a, \tag{4.21}$$

and taking the logarithmic derivative on both sides of (4.20), we have

$$\begin{aligned}
\frac{d \ln m_l}{ds} &= (\ln r_l)_s - (\ln r_{l+1})_s - \frac{8}{(4 - \delta_l^2)\delta_l} \frac{d\delta_l}{ds} - \frac{d}{ds} \ln(\delta_l + \delta_{l-1}) \\
&= (\ln r_l)_s - (\ln r_{l+1})_s - 2 \frac{u_{l+1} - u_l}{\delta_l} - \frac{d}{ds} \ln(\delta_l + \delta_{l-1}) \quad (\text{by (4.19)})
\end{aligned}$$

$$\begin{aligned}
&= -\frac{u_l - u_{l-1}}{\delta_{l-1}} - \frac{\delta_{l-1}}{2} \left( \frac{u_l + u_{l-1}}{2} - a \right) + \frac{u_{l+1} - u_l}{\delta_l} + \frac{\delta_l}{2} \left( \frac{u_{l+1} + u_l}{2} - a \right) \\
&\quad - 2 \frac{u_{l+1} - u_l}{\delta_l} - \frac{d}{ds} \ln(\delta_l + \delta_{l-1}) \quad (\text{by (4.18)}) \\
&= -\frac{u_l - u_{l-1}}{\delta_{l-1}} - \frac{\delta_{l-1}}{2} \left( \frac{u_l + u_{l-1}}{2} - a \right) - \frac{u_{l+1} - u_l}{\delta_l} + \frac{\delta_l}{2} \left( \frac{u_{l+1} + u_l}{2} - a \right) \\
&\quad - \frac{1}{\delta_l + \delta_{l-1}} \left( (u_{l+1} - u_{l-1}) - \frac{\delta_l^2(u_{l+1} - u_l) + \delta_{l-1}^2(u_l - u_{l-1})}{4} \right). \quad (4.22)
\end{aligned}$$

By defining forward difference and average operators

$$\Delta u_l = \frac{u_{l+1} - u_l}{\delta_l}, \quad Mu_l = \frac{u_l + u_{l-1}}{2},$$

we can summarize what we have deduced into the following theorem.

**Theorem 4.2.** *The semi-discrete Degasperis-Procesi equation*

$$\begin{cases} \frac{d \ln m_l}{ds} = -2M\Delta u_l - \frac{M(\delta_l \Delta u_l)}{M\delta_l} + \frac{\delta_l(Mu_{l+1} - a) - \delta_{l-1}(Mu_l - a)}{2} + \frac{M(\delta_l^2(u_{l+1} - u_l))}{4M\delta_l}, \\ \frac{d \delta_l}{ds} = \left(1 - \frac{\delta_l^2}{4}\right)(u_{l+1} - u_l) \\ m_l = -\frac{\Delta u_l - \Delta u_{l-1}}{M\delta_l} + \frac{M(\delta_l(Mu_l))}{M\delta_l} - a, \end{cases} \quad (4.23)$$

is determined from the following equations

$$\begin{cases} \left( \frac{1}{a+b} D_s - 1 \right) g_{l+1} \cdot f_l = \left( \frac{1}{a-b} D_s - 1 \right) g_l \cdot f_{l+1}, \\ g_l f_l = c' G_l, \\ (D_s - 2b) F_{l+1} \cdot F_l = -2b G_l^2, \\ (a-b) g_{l+1} f_l - (a+b) g_l f_{l+1} = -2bc' F_{l+1}. \end{cases}$$

through discrete hodograph transformation

$$\delta_l = 2 \frac{(a-b)g_{l+1}f_l - (a+b)g_l f_{l+1}}{(a-b)g_{l+1}f_l + (a+b)g_l f_{l+1}}, \quad t = s$$

and dependent variable transformation

$$u_l = \left( \ln \frac{g_l}{f_l} \right)_s$$

Let us consider the continuous limit when  $b \rightarrow 0$ . The dependent variable  $u$  is a function of  $l$  and  $s$ . Meanwhile, we regard it as a function of  $x$  and  $t$ , where  $x$  is the space coordinate at  $l$ -th lattice point and  $t$  is the time, defined by

$$x = x_0 + \sum_{j=0}^{l-1} \delta_j, \quad t = s$$

Then in the continuous limit,  $b \rightarrow 0$  ( $\delta_l \rightarrow 0$ ), we have

$$\begin{aligned} 2M\Delta u_l &= \frac{u_{l+1} - u_l}{\delta_l} + \frac{u_l - u_{l-1}}{\delta_{l-1}} \rightarrow 2u_x, \quad \frac{M(\delta_l \Delta u_l)}{M\delta_l} = \frac{u_{l+1} - u_{l-1}}{\delta_l + \delta_{l-1}} \rightarrow u_x, \\ \frac{\delta_{l-1}}{2}(Mu_l - a) &\rightarrow 0, \quad \frac{M(\delta_l^2(u_{l+1} - u_l))}{M\delta_l} \rightarrow 0, \\ m_l &= -\frac{(\Delta u_l - \Delta u_{l-1})}{M\delta_l} + \frac{M(\delta_l(Mu_l))}{M\delta_l} - a \rightarrow m = u - u_{xx} - a. \end{aligned}$$

Moreover, since

$$\frac{\partial x}{\partial s} = \frac{\partial x_0}{\partial s} + \sum_{j=0}^{l-1} \frac{\partial \delta_j}{\partial s} = \frac{\partial x_0}{\partial s} + \sum_{j=0}^{l-1} \left(1 - \frac{\delta_j^2}{4}\right) (u_{j+1} - u_j) \rightarrow u,$$

we then have

$$\partial_s = \partial_t + \frac{\partial x}{\partial s} \partial_x \rightarrow \partial_t + u \partial_x.$$

Consequently, the third equation in (4.23) converges to  $m = u - u_{xx} - a$ . Whereas the first equation in (4.23) converges to

$$(\partial_t + u \partial_x)m = -3mu_x, \quad (4.24)$$

which is exactly the Degasperis-Procesi equation (1.2). Based on the results in previous section, we can provide  $N$ -soliton solution to the semi-discrete Degasperis-Procesi equation

**Theorem 4.3.** *The  $N$ -soliton solution to the semi-discrete analogue of the Degasperis-Procesi equation (4.23) takes the following parametric form*

$$u_l = \left( \ln \frac{g_l}{f_l} \right)_s, \quad \delta_l = 2 \frac{(a-b)g_{l+1}f_l - (a+b)g_l f_{l+1}}{(a-b)g_{l+1}f_l + (a+b)g_l f_{l+1}}, \quad (4.25)$$

where  $g_l = f_{1l}$ ,  $f_l = f_{0l}$  with pfaffian  $f_{kl}$  defined by

$$f_{kl} = \text{Pf}(1, 2, \dots, 2N)_{kl}, \quad (4.26)$$

whose elements are

$$(i, j)_{kl} = c_{i,j} + \frac{p_i - p_j}{p_i + p_j} \varphi_i^{(0)}(k, l) \varphi_j^{(0)}(k, l), \quad (4.27)$$

$$\varphi_i^{(n)}(k, l) = p_i^n \left( \frac{1 + ap_i}{1 - ap_i} \right)^k \left( \frac{1 + bp_i}{1 - bp_i} \right)^l e^{p_i^{-1}s + \xi_{i0}} \quad (4.28)$$

under the reduction condition ( $i = 1, 2, \dots, N$ )

$$p_i(1 - a^2 p_i^2)(1 - b^2 p_{2N+1-i}^2) + p_{2N+1-i}(1 - a^2 p_{2N+1-i}^2)(1 - b^2 p_i^2) = 0. \quad (4.29)$$

In the last, we calculate the  $\tau$ -functions for one- and two-soliton solutions, and compared them with the ones of the DP equation (1.2).

**One-soliton:** For  $N = 1$ , we have

$$g_l = \text{Pf}(1, 2)_{10} = c_{1,2} + \frac{p_1 - p_2}{p_1 + p_2} \varphi_1^{(0)}(1, l) \varphi_2^{(0)}(1, l) \quad (4.30)$$

$$\propto 1 + e^{\xi_1(l) + \xi_2(l) + \phi_1}, \quad (4.31)$$

$$f_l = \text{Pf}(1, 2)_{00} = c_{1,2} + \frac{p_1 - p_2}{p_1 + p_2} \phi_1^{(0)}(0, l) \phi_2^{(0)}(0, l) \quad (4.32)$$

$$\propto 1 + e^{\xi_1(l) + \xi_2(l) - \phi_1}, \quad (4.33)$$

where

$$e^{\xi_i(l)} = \left( \frac{1 + bp_i}{1 - bp_i} \right)^l e^{p_i^{-1}s + \xi_{i0}}, \quad (i = 1, 2), \quad e^{\phi_1} = \sqrt{\frac{(1 + ap_1)(1 + ap_2)}{(1 - ap_1)(1 - ap_2)}}.$$

Here  $p_1, p_2 = p_1^*$  are two parameters related by a constraint

$$p_1(1 - a^2 p_1^2)(1 - b^2 p_2^2) + p_2(1 - a^2 p_2^2)(1 - b^2 p_1^2) = 0. \quad (4.34)$$

Let  $p_1 = A_1 e^{i\theta_1}$ ,  $p_2 = A_1 e^{-i\theta_1}$ , and  $p_1 + p_2 = k_1$ , we then have

$$1 - a^2 k_1^2 + (3a^2 - b^2)A_1^2 + a^2 b^2 A_1^4 = 0, \quad (4.35)$$

from which  $A^2$  can be solved as

$$A_1^2 = \frac{\sqrt{(3a^2 - b^2)^2 - 4a^2 b^2(1 - a^2 k_1^2)} - (3a^2 - b^2)}{2a^2 b^2}. \quad (4.36)$$

In the continuous limit of  $b \rightarrow 0$  and  $a = -1$ , a simple calculation gives

$$A_1^2 \rightarrow \frac{k_1^2 - 1}{3}, \quad (4.37)$$

and

$$e^{\xi_1(l) + \xi_2(l)} \rightarrow e^{-2bl(p_1 + p_2) + (p_1^{-1} + p_2^{-1})s + \eta_{10}} \rightarrow e^{k_1 y + \frac{3k_1}{k_1^2 - 1}s + \eta_{10}} \quad (4.38)$$

by letting  $y = -2bl$ . Therefore, the one-soliton solution of the semi-discrete DP equation converges to the one-soliton solution given in [16, 17, 18].

### Two-soliton: Two-soliton

For  $N = 2$ , by assuming  $\phi_{ij}^{(0)}(k, l) = \phi_i^{(0)}(k, l) \phi_j^{(0)}(k, l)$ , we have

$$\begin{aligned} g_l &= \text{Pf}(1, 2, 3, 4)_{10} = \text{Pf}(1, 2)_{10} \text{Pf}(3, 4)_{10} - \text{Pf}(1, 3)_{10} \text{Pf}(2, 4)_{10} + \text{Pf}(1, 4)_{10} \text{Pf}(2, 3)_{10} \\ &= \frac{p_1 - p_2}{p_1 + p_2} \phi_{12}^{(0)}(1, l) \times \frac{p_3 - p_4}{p_3 + p_4} \phi_{34}^{(0)}(1, l) - \frac{p_1 - p_3}{p_1 + p_3} \phi_{13}^{(0)}(1, l) \times \frac{p_2 - p_4}{p_2 + p_4} \phi_{24}^{(0)}(1, l) \\ &\quad + \left( c_{14} + \frac{p_1 - p_4}{p_1 + p_4} \phi_{14}^{(0)}(1, l) \right) \left( c_{23} + \frac{p_2 - p_3}{p_2 + p_3} \phi_{23}^{(0)}(1, l) \right) \\ &\propto 1 + e^{\xi_1(l) + \xi_4(l) + \phi_1 + \gamma_1} + e^{\xi_2(l) + \xi_3(l) + \phi_2 + \gamma_2} + b_{12} e^{\sum_{j=1}^4 \xi_j(l) + \phi_1 + \phi_2 + \gamma_1 + \gamma_2}, \end{aligned} \quad (4.39)$$

$$f_l = \text{Pf}(1, 2, 3, 4)_{00}$$

$$\propto 1 + e^{\xi_1(l) + \xi_4(l) - \phi_1 + \gamma_1} + e^{\xi_2(l) + \xi_3(l) - \phi_2 + \gamma_2} + b_{12} e^{\sum_{j=1}^4 \xi_j(l) - \phi_1 - \phi_2 + \gamma_1 + \gamma_2}, \quad (4.40)$$

under the condition

$$p_i(1 - a^2 p_i^2)(1 - b^2 p_{2N+1-i}^2) + p_{2N+1-i}(1 - a^2 p_{2N+1-i}^2)(1 - b^2 p_i^2) = 0 \quad (i = 1, 2). \quad (4.41)$$

Here  $c_{14} = c_{23} = 1$ ,  $e^{\phi_1} = \sqrt{\frac{(1 + ap_1)(1 + ap_4)}{(1 - ap_1)(1 - ap_4)}}$ ,  $e^{\phi_2} = \sqrt{\frac{(1 + ap_2)(1 + ap_3)}{(1 - ap_2)(1 - ap_3)}}$ ,  $e^{\gamma_1} = \frac{p_1 - p_4}{p_1 + p_4}$ ,  $e^{\gamma_2} = \frac{p_2 - p_3}{p_2 + p_3}$ , and  $b_{12} = \frac{(p_1 - p_2)(p_1 - p_3)(p_4 - p_2)(p_4 - p_3)}{(p_1 + p_2)(p_1 + p_3)(p_4 + p_2)(p_4 + p_3)}$ . In the continuous limit  $b \rightarrow 0$ , we can show that the two-soliton solutions for semi-discrete DP equation converge to the two-soliton solutions of the DP equation found in [16, 17, 18] by letting  $p_1 + p_4 = k_1$ ,  $p_2 + p_3 = k_2$ .

## 5. Conclusion and further topics

In the present paper, we firstly review a set of equations which drive the DP equation through a dependent variable transformation and a hodograph transformation. Then by constructing the integrable semi-discrete analogues of these equations including bilinear equations and by defining a discrete hodograph transformation, an integrable semi-discrete DP equation was proposed.

Similar to what we have done for the CH equation [30], the short pulse equation [20] and a coupled short pulse equation [22], it deserves exploring a problem of using the proposed semi-discrete DP equation as a self-adaptive moving mesh scheme for the numerical simulation of the DP equation.

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## Appendix A

$$\begin{aligned}
 \partial_s f_{kl} &= \partial_s \text{Pf}_{1 \leq i < j \leq 2N} \left( (i, j)_{kl} \right) \\
 &= \text{Pf} \begin{pmatrix} (i, j)_{kl} & \varphi_i^{(-1)}(k, l) & \varphi_i^{(0)}(k, l) \\ & & 0 \end{pmatrix} \\
 &= (1, 2, \dots, 2N, d_{-1}, d_0)_{kl}
 \end{aligned} \tag{A.1}$$

$$\begin{aligned}
 f_{k+1, l} &= \text{Pf}_{1 \leq i < j \leq 2N} \left( (i, j)_{kl} + \varphi_i^{(0)}(k+1, l) \varphi_j^{(0)}(k, l) - \varphi_i^{(0)}(k, l) \varphi_j^{(0)}(k+1, l) \right) \\
 &= \text{Pf} \begin{pmatrix} (i, j)_{kl} & \varphi_i^{(0)}(k, l) & \varphi_i^{(0)}(k+1, l) \\ & & 1 \end{pmatrix} \\
 &= (1, 2, \dots, 2N, d_0, d^k)_{kl}
 \end{aligned} \tag{A.2}$$

$$\begin{aligned}
 (\partial_s - a) f_{k+1, l} &= (\partial_s - a) \text{Pf} \begin{pmatrix} (i, j)_{kl} & \varphi_i^{(0)}(k, l) & \varphi_i^{(0)}(k+1, l) \\ & & 1 \end{pmatrix} \\
 &= \text{Pf} \begin{pmatrix} (i, j)_{kl} & \varphi_i^{(-1)}(k, l) & \varphi_i^{(0)}(k, l) & \varphi_i^{(0)}(k, l) & \varphi_i^{(0)}(k+1, l) \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 1 \end{pmatrix} \\
 &\quad + \text{Pf} \begin{pmatrix} (i, j)_{kl} & \partial_s \varphi_i^{(0)}(k, l) & \varphi_i^{(0)}(k+1, l) \\ & & 0 \end{pmatrix}
 \end{aligned}$$



$$\begin{aligned}
& + \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & (\partial_s - a)\varphi_i^{(0)}(k+1,l) \\ & & -a \end{pmatrix} \\
& = \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) & \varphi_i^{(0)}(k,l) & 0 & \varphi_i^{(0)}(k+1,l) \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 1 \end{pmatrix} \\
& + \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) & \varphi_i^{(0)}(k+1,l) \\ & & 0 \end{pmatrix} \\
& + \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & \varphi_i^{(-1)}(k,l) + a\varphi_i^{(0)}(k,l) \\ & & -a \end{pmatrix} \\
& = \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) & \varphi_i^{(0)}(k,l) \\ & & 0 \end{pmatrix} \\
& + \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) & \varphi_i^{(0)}(k+1,l) \\ & & 0 \end{pmatrix} + \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & \varphi_i^{(-1)}(k,l) \\ & & -a \end{pmatrix} \\
& = \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) & \varphi_i^{(0)}(k+1,l) \\ & & -a \end{pmatrix} \\
& = (1, 2, \dots, 2N, d_{-1}, d^k)_{kl} \tag{A.3}
\end{aligned}$$

Similarly, we have

$$f_{k,l+1} = (1, 2, \dots, 2N, d_0, d^l)_{kl}. \tag{A.4}$$

$$(\partial_s - b)f_{k,l+1} = (1, 2, \dots, 2N, d_{-1}, d^l)_{kl}. \tag{A.5}$$

$$\begin{aligned}
\frac{a-b}{a+b}f_{k+1,l+1} &= \frac{a-b}{a+b} \text{Pf} \begin{pmatrix} (i,j)_{k,l+1} & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l+1) \\ & & 1 \end{pmatrix} \\
&= \frac{a-b}{a+b} \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l+1) \\ & & 1 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 1 \end{pmatrix} \\
&= \frac{a-b}{a+b} \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l+1) - \varphi_i^{(0)}(k,l) \\ & & 1 & 0 & 0 \\ & & & 0 & 1 \\ & & & & 1 \end{pmatrix} \\
&= \frac{a-b}{a+b} \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l) & 0 & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l+1) - \varphi_i^{(0)}(k,l) \\ & & 1 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \frac{a-b}{a+b} \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l+1) - \varphi_i^{(0)}(k,l) \\ & & 1 \end{pmatrix} \\
&= \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l+1) & \frac{a-b}{a+b}(\varphi_i^{(0)}(k+1,l+1) - \varphi_i^{(0)}(k,l)) \\ & & \frac{a-b}{a+b} \end{pmatrix} \\
&= \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l) - \varphi_i^{(0)}(k,l+1) \\ & & \frac{a-b}{a+b} \end{pmatrix} \\
&= \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l) \\ & & \frac{a-b}{a+b} \end{pmatrix} \\
&= (1, 2, \dots, 2N, d^l, d^k)_{kl}
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
(\partial_s - a - b) \frac{a-b}{a+b} f_{k+1,l+1} &= (\partial_s - a - b) \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l) \\ & & \frac{a-b}{a+b} \end{pmatrix} \\
&= \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l) \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & \frac{a-b}{a+b} \end{pmatrix} \\
&\quad + \text{Pf} \begin{pmatrix} (i,j)_{kl} & (\partial_s - b) \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l) \\ & & -b \frac{a-b}{a+b} \end{pmatrix} \\
&\quad + \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l+1) & (\partial_s - a) \varphi_i^{(0)}(k+1,l) \\ & & -a \frac{a-b}{a+b} \end{pmatrix} \\
&= \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l) \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \\ & & & & \frac{a-b}{a+b} \end{pmatrix} \\
&\quad + \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) + b \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k+1,l) \\ & & -b \frac{a-b}{a+b} \end{pmatrix} \\
&\quad + \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(-1)}(k,l) + a \varphi_i^{(0)}(k,l) \\ & & -a \frac{a-b}{a+b} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l) \\ & & 0 & 0 & -a \\ & & & 0 & \frac{1}{a-b} \\ & & & & \frac{a-b}{a+b} \end{pmatrix} \\
&+ \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) + b\varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k+1,l) \\ & & -a\frac{a-b}{a+b} - b\frac{a-b}{a+b} \end{pmatrix} \\
&= \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l) \\ & & 0 & 0 & -a \\ & & & 0 & \frac{1}{a-b} \\ & & & & \frac{a-b}{a+b} \end{pmatrix} \\
&+ \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) + b\varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k+1,l) \\ & & -a+b \end{pmatrix} \\
&= \text{Pf} \begin{pmatrix} (i,j)_{kl} & \varphi_i^{(-1)}(k,l) & \varphi_i^{(0)}(k,l) & \varphi_i^{(0)}(k,l+1) & \varphi_i^{(0)}(k+1,l) \\ & & 0 & -b & -a \\ & & & 1 & \frac{1}{a-b} \\ & & & & \frac{a-b}{a+b} \end{pmatrix} \\
&= (1, 2, \dots, 2N, d_{-1}, d_0, d^l, d^k)_{kl}
\end{aligned} \tag{A.7}$$

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